Universal formula for the mean first passage time in planar domains

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We present a general exact formula for the mean first passage time (MFPT), $T(x_0)$, from a fixed point x_0 inside a planar simply connected domain Ω to a connected escape region Γ on the boundary $\partial\Omega$ [1]. The underlying mixed Dirichlet-Neumann boundary value problem is conformally mapped onto the unit disk, solved exactly, and mapped back (Fig. 1). The resulting formula (1) for the MFPT is valid for an arbitrary space-dependent diffusion coefficient $D(x)$, while the leading logarithmic term in Eq. (3) is explicit, simple, and remarkably universal. In contrast to earlier works [2], we show that the natural small parameter of the problem is the harmonic measure ω of the escape region, not its perimeter. The conventional scaling of the MFPT with the area of the domain is altered when diffusing particles are released near the escape region. These findings change the current view of escape problems and related chemical or biochemical kinetics in complex, multiscale, porous or fractal domains, while the fundamental relation to the harmonic measure opens new ways of computing and interpreting MFPTs.

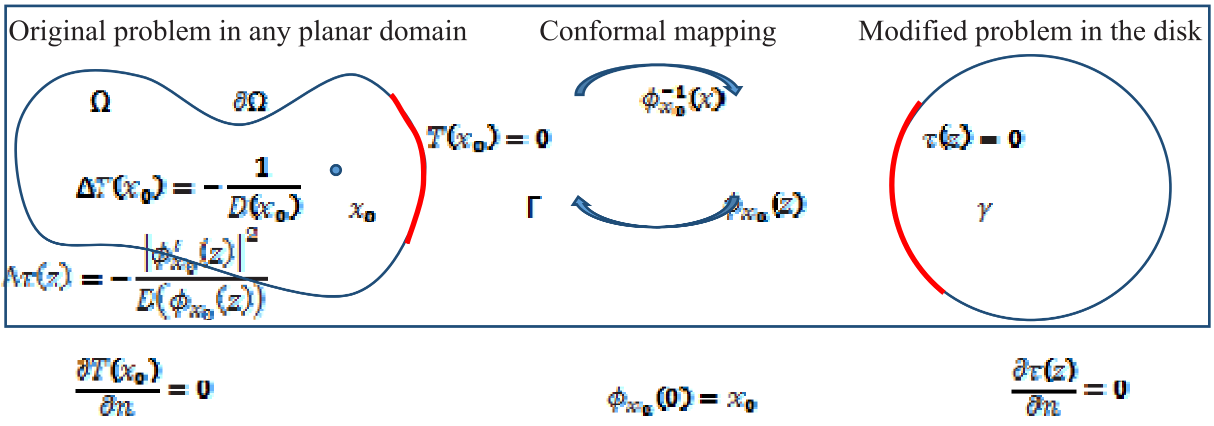

Figure 5: An arbitrary simply connected domain Ω (left) can be mapped onto the unit disk (right) by a conformal mapping $\phi_{x_0}(z)$ such that the origin of the disk is mapped onto the starting point x_0. The escape region Γ (in red) is mapped onto the arc $\gamma = (-\pi\omega, \pi\omega)$, where $\omega = \omega_{x_0}(\Gamma)$ is the harmonic measure of Γ seen from x_0.

The universal formula for the MFPT reads

$$T(x_0) = \int_\Omega \frac{dx}{D(x)} \left(-\frac{\ln|\phi_{x_0}^{-1}(x)|}{2\pi} + W_\omega\left(\phi_{x_0}^{-1}(x)\right) \right)$$

(1)

where

$$W_\omega(z) = \frac{1}{\pi} \ln \left(\frac{\left| 1 - z + \sqrt{(1 - z\, e^{i\pi\omega})(1 - z\, e^{-i\pi\omega})} \right|}{2 \sin\left(\frac{\pi\omega}{2}\right)} \right)$$

(2)

The asymptotic expansion of Eq. (1) over the natural small parameter, the harmonic measure ω , yields

$$T(x_0) = \frac{|\Omega|}{\pi D_h} \ln\left(\frac{1}{\omega}\right) + V_0(x_0) + V_2(x_0)\omega^2 + O(\omega^4)$$

(3)

where the explicit formulas for the coefficients $V_0(x_0)$, $V_2(x_0), \ldots$ are given in [1], and $D_h = \left(\frac{1}{|\Omega|} \int_\Omega \frac{dx}{D(x)} \right)^{-1}$ is the harmonic mean of the diffusion coefficient $D(x)$. The leading logarithmic term substitutes the conventional scaling $\frac{|\Omega|}{\pi D} \ln\left(\frac{1}{\varepsilon}\right)$ with the normalized perimeter ε [2].

References

[1] D. S. Grebenkov: *Universal formula for the mean first passage time in planar domains,* Phys. Rev. Lett. **117**, 260201 (2016).

[2] D. Holcman and Z. Schuss, *The Narrow Escape Problem*, SIAM Rev. **56**, 213-257 (2014).