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Weak ergodicity breaking for anomalous diffusion

Eli Barkai

Physics Department, Bar Ilan University, Ramat Gan, Israel
barkaie@biu.ac.il

We briefly review basic random walk approaches to anomalous diffusion: the sub-diffusive continuous time random walk, super-diffusive Lévy walks, and Lévy flights. The corresponding fractional diffusion equations are presented, and we then turn to two physical applications. For cold atoms diffusing in an optical lattice, we derive the fractional space diffusion equation from the semiclassical description of Sisyphus cooling [1]. Depending on optical lattice depth, we show that the dynamics of atoms is described by normal diffusion, Lévy flights with cutoffs, or Richardson-Obukhov dynamics, .i.e. $x^2 \sim t^3$ found in the field of turbulence. This rich phase diagram is related to the peculiar friction force induced by the laser field. Unlike Stokes friction which increases with velocity, in our system the friction decays like $1/v$ for large velocities, thus asymptotically the system is frictionless, leading to fat tailed dynamics. Comparison to experiment, indicates that open questions still remain in this field [2].

The second theme we address, is the question of ergodicity in anomalous processes. For Brownian motion the ensemble averaged mean squared displacement (MSD) $\langle x^2 \rangle$ is identical to the time averaged MSD

$$\overline{\delta^2} = \frac{\int_0^{t-\Delta} [x(t'+\Delta) - x(t')]^2 \, \mathrm{d}t'}{t-\Delta} \tag{1}$$

in the limit where the measurement time t is much longer than the lag time Δ. Recent works showed that for the above mentioned models of anomalous diffusion, and for a wide variety of anomalous diffusion processes of single molecules in the cell environment, we get $\overline{\delta^2} \neq \langle x^2 \rangle$. This being in complete contrast to normal diffusion, and it indicates that ergodicity in the MSD sense cannot be taken for granted [3]. We quantify deviations from ergodic behaviour, showing that time averaged transport coefficients, remain random with a well defined probability density [4, 5]. For the sub-diffusive continuous time random walk, the time averaged MSD exhibits ageing, namely it crucially depends on both the measurement time t and lag time Δ, in agreement with recent single molecule tracking experiments [6], on sub-diffusive Kv2.1 potassium channel in the plasma membrane.

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