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The Open-Access Journal for the Basic Principles of Diffusion Theory, Experiment and Application

Langevin-Vladimirsky approach to Brownian motion with memory

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(received 14 July 2011, accepted 25 July 2011)

Abstract

A number of random processes in various fields of science is described by phenomenological equations of motion containing stochastic forces, the best known example being the Langevin equation (LE) for the Brownian motion (BM) of particles. Long ago Vladimirsky (1942) in a little known paper proposed a simple method for solving such equations. The method, based on the classical Gibbs statistics, consists in converting the stochastic LE into a deterministic equation for the mean square displacement of the particle, and is applicable to linear equations with any kind of memory in the dynamics of the system. This approach can be effectively used in solving many of the problems currently considered in the literature. We apply it to the description of the BM when the noise driving the particle is exponentially correlated in time. The problem of the hydrodynamic BM of a charged particle in an external magnetic field is also solved.

Keywords

Generalized Langevin equation, colored stochastic force, hydrodynamic Brownian motion

1. Introduction

During the past decades, a continuously increasing interest in various fields of science and technology is observed to diffusion processes in systems exhibiting different kinds of memory [1]. A variety of physical, chemical, biological and other natural phenomena can be adequately described by random processes that at long times show a behavior, which is characterized by a nonlinear dependence of the variance of these processes on time. Then the famous Einstein formula [2] for the long-time behavior of mean square displacement (MSD) of the Brownian particle (BP) changes to $\langle[\mathbf{r}(t) - \mathbf{r}(0)]^2\rangle \sim t^\alpha$ with the constant α that can differ from 1. Mathematically, such "anomalous" diffusion is described by fractional differential equations or Volterra-type integro-differential equations with colored stochastic forces driving the particle [3, 4]. These equations generalize the well-known Langevin equation (LE) [5] designed for the description of memory-less Brownian motion (BM). In some cases they at t

$\to \infty$ lead to "normal diffusion" with $\alpha = 1$ but due to the memory in the system there are important corrections to the Einstein formula at finite times. In the present work we use a very simple way of solution of the generalized LE (GLE) for the BM with memory. The method, proposed long ago by Vladimirsky in a hardly accessible and little known work [6], can be used for the description of the BM in linear classical systems with any kind of memory. Using it, the general properties of anomalous diffusion can be easily derived. We consider a special kind of the memory in the system, which is associated with the driving stochastic force represented by the Ornstein-Uhlenbeck process. The exact solution of the GLE is given for this case. The effectiveness of the method is shown also on the example of the hydrodynamic BM of a charged particle moving in an incompressible liquid along and across an external magnetic field.

2. Efficient rule for solving the Langevin equation

In Ref. [6], Vladimirsky considered a method for the evaluation of the averaged products of two thermally fluctuating quantities related to different moments of time. The difficult task of the calculation of such means using the Gibbs method was reduced to a much simpler solution of equations of motion. The obtained rules of finding the time correlation functions are thus consistent with the general principles of statistical mechanics and simultaneously with the linear phenomenological equations of motion for the studied quantities. For the purposes of the present paper, let us consider the LE for the coordinate x of the BP. If we are interested in finding the MSD of the particle, $\xi(t) = \langle \Delta x^2(t) \rangle = \langle [x(t) - x(0)]^2 \rangle$, according to the Vladimirsky rule we have merely to substitute $x(t)$ in the LE by $\xi(t)$ and replace the stochastic force driving the particle with the constant "force" $f = 2k_BT$, where k_B is the Boltzmann constant and T the temperature. Of course, having the dimension of energy, f is not a true force; it only plays this role in the equation for the particle "position" $\xi(t)$. This fictitious force begins to act on the particle at the time $t = 0$. Up to this moment the particle is nonmoving so that the equation of motion must be solved with the initial conditions $\xi(0) = V(0) = 0$, where we have introduced the "velocity" $V(t) = \mathrm{d}\xi(t)/\mathrm{d}t$. It is important that at the initial moment the whole studied system is assumed to be in the thermodynamically equilibrium state. From the point of view of the macroscopic equations of motion such initial state corresponds to the long-lasting rest of the system. Thus the values of all the parameters entering the equations of motion are specified for any $t \le 0$. This allows one to study not only equations for $x(t)$ in the form of differential equations (that require only the specifications of the initial values of the quantities) but also any Volterra-type integral equations. Such equations can describe systems for which the whole history of its motion is important. The discussed method is not limited to a concrete law of the aftereffect. Its advantage is that it is applicable to any kind of the memory in the studied (linear) system and the only requirement is the linearity and existence of the unique solution at the given initial conditions.

Let us illustrate the described rule by a simple example of a damped harmonic oscillator under the influence of the thermal Langevin force $f(t)$. The motion of the oscillator is described by the equation for the position $x(t)$, $m\ddot{x} + \gamma\dot{x} + m\omega_0^2 x = f(t)$ (the parameters m, γ and ω can be regarded as the mass of a particle, the friction constant, and the oscillator frequency, respectively). By the same way as in the original Langevin paper [5] (see also [4]), assuming in the averaging that x, $\dot{x}$ and f are statistically independent, and with the help of the equipartition theorem $m\langle \dot{x}^2 \rangle = k_BT$, the original equation can be for stationary random processes easily rewritten in the form $m\ddot{\xi} + \gamma\dot{\xi} + m\omega_0^2\xi = 2k_BT$, with the evident initial conditions $\xi(0) = \dot{\xi}(0) = 0$. Analogously, more complicated phenomenological (but linear) stochastic equations can be converted to deterministic ones, e.g., in the case when the friction force $-\gamma\dot{x}$ is replaced by a convolution of the time dependent memory kernel Γ(t) and the particle velocity $\dot{x}(t)$. This corresponds to the GLE considered in the next section.

3. Solving the generalized Langevin equation

So, already in 1942 [6] an effective approach to various problems that are now extensively studied in the literature was available. Many non-Markovian processes are described by the GLE [1, 3, 4, 7, 8]

$$M\dot{\upsilon}+\int_0^t \Gamma\left(t-t'\right)\upsilon\left(t'\right)\mathrm{d}t'=\eta\left(t\right), \tag{1}$$

where, for the BM of particles, M could be the mass of the BP with the velocity $\upsilon(t)=\dot{x}(t)$. The memory in the system is described by the kernel Γ(t), and $\eta(t)$ is a stochastic force with zero mean. According to the fluctuation-dissipation theorem the condition $\langle\eta(0)\eta(t)\rangle$ = $k_B T\Gamma(t)$ at $t > 0$ must be satisfied [3, 7 - 9]. One is usually interested in finding the MSD of the particle, $\xi(t)=\int_0^t V(\tau)\mathrm{d}\tau$, its time-dependent diffusion coefficient $D(t)$ = $V(t)/2$, or the velocity autocorrelation function (VAF) $\Phi(t)=\dot{D}(t)$ [10]. The initial conditions for $\xi(t)$ and $V(t)$ have been discussed above. Obviously, also the condition $\dot{V}(0)=2k_B T/M$ must hold. It follows from the equation that concretizes the Vladimirsky rule for this special case,

$$M\dot{V}+\int_0^t \Gamma\left(t-t'\right)V\left(t'\right)\mathrm{d}t'=2k_B T\,. \tag{2}$$

Taking the Laplace transformation Λ of this equation with the use of the initial conditions, it can be easily solved for the specific memory kernels considered in the literature. Also the known general properties of the MSD and other time correlation functions [11, 12] can be obtained in this way, which proves the Vladimirsky rule for equation (1). The long-time properties of the MSD can be related to the behavior of the Laplace transform of the kernel Γ, $\tilde{\Gamma}(s)=\Lambda\{\Gamma(t)\}$ at small s [11, 13]. At short times, equation (2) describes the ballistic motion, $\xi(t)\approx k_B T t^2/M$ as $t\to 0$, independently of a concrete form of Γ(t).

As an example, let us specify the memory kernel. We chose Γ(t) in the Ornstein-Uhlenbeck form $\Gamma(t) = (\gamma^2/m)\exp(-\gamma t/m)$. Then the force $\eta(t)$ = $m\dot{u}(t)$ corresponds to the solution of the usual LE $m\dot{u}(t)+\gamma u(t)=f(t)$. Here $u(t)$ can be the velocity of a BP, γ is a constant friction coefficient, and $f(t)$ the stochastic (white noise) force. The motion of the BPs with the mass M can be thus interpreted as being induced by the force η caused by the particles of mass m.

The Laplace transformation of (2) yields

$$\tilde{V}(s)=\frac{2k_B T}{Ms}\left(s+\frac{\gamma}{m}\right)\left(s^2+\frac{\gamma}{m}s+\frac{\gamma^2}{mM}\right)^{-1} \tag{3}$$

The denominator in this expression can be given the form (s - α_1)(s - α_2), $\alpha_{1,2}=(\gamma/2m)(1\mp\sqrt{1-4m/M})$, so that the solution is expressed as

$$\tilde{V}(s) = \frac{2k_BT}{M}\frac{1}{\alpha_2-\alpha_1}\left(1+\frac{\gamma}{m}\frac{1}{s}\right)\left(\frac{1}{s-\alpha_2}-\frac{1}{s-\alpha_1}\right). \tag{4}$$

Using the tables of Laplace transforms [14] we get

$$V(t) = \frac{2k_BT}{M}\left\{\frac{\gamma}{m\alpha_1\alpha_2}+\frac{1}{\alpha_2-\alpha_1}\left[\left(1+\frac{\gamma}{m\alpha_2}\right)\exp\left(\alpha_2 t\right)-\left(1+\frac{\gamma}{m\alpha_1}\right)\exp\left(\alpha_1 t\right)\right]\right\}. \tag{5}$$

The MSD is obtained by simple integration of this equation. It can be used that $\alpha_1\alpha_2 = \gamma^2/(mM)$; it is then seen that the first term in {} is the Einstein long-time limit $2k_BT/\gamma$. At short times we get the expected ballistic motion. If we denote $\mu = \sqrt{1-4m/M}$, $D(t)$ can be given a compact form

$$D(t) = \frac{k_BT}{\gamma}\left\{1-\frac{1}{4\mu}\exp(-\gamma t/2m)\left[(1+\mu)^2\exp(\gamma\mu t/2m)-(1-\mu)^2\exp(-\gamma\mu t/2m)\right]\right\}. \tag{6}$$

If we forget that the motion of the particles was interpreted as induced by particles with smaller mass m, an interesting result follows from equation (5) in the case when $M < 4m$, i.e. when the roots α are complex. Then the solution describes damped oscillations; *e.g.* for $D(t)$ we have

$$D(t) = \frac{k_BT}{\gamma}\left\{1-\exp\left(-\frac{\gamma}{2m}t\right)\times\left[\cos\left(\frac{\gamma t}{2m}\sqrt{\frac{4m}{M}-1}\right)-\frac{2m/M-1}{\sqrt{4m/M-1}}\sin\left(\frac{\gamma t}{2m}\sqrt{\frac{4m}{M}-1}\right)\right]\right\}$$
$$\approx \frac{k_BT}{\gamma}\left\{1+\exp(-\gamma t/2m)\left[\sqrt{\frac{m}{M}}\sin\left(\frac{\gamma t}{\sqrt{mM}}\right)-\cos\left(\frac{\gamma t}{\sqrt{mM}}\right)\right]\right\}\ (\text{if } M << 4m)\,. \tag{7}$$

The special case of constant memory kernel Γ [15] follows from here when $\gamma/m \to 0$ ($\Gamma = \gamma^2/m$ = const):

$$D(t) \approx \frac{k_BT}{\sqrt{\Gamma M}}\sin\left(\sqrt{\frac{\Gamma}{M}}t\right). \tag{8}$$

Note that the results (5) and (6) (or the corresponding VAF) can be obtained also from the solution of the Fokker-Planck equation (FPE), associated with the GLE (1). The FPE possesses the distribution functions of fluctuating macroscopic variables, which in many cases give a more convenient probabilistic description of the studied processes. As distinct from the used approach that does not explicitly explore the correlation properties of the random force $\eta(t)$, building the FPE requires the knowledge of these properties [16]. The generalized FPE corresponding to the GLE (1) with the memory kernel $\Gamma(t - t')$ and to equation (2) (with the initial instant of time $t_0 = 0$) was studied in Ref. [17]. A more general FPE was obtained in the remarkable work [18]. The FPE obtained in that work is associated with the system of Langevin equations for multidimensional non-stationary process with the memory kernels $\Gamma_{\alpha\beta}(t, t')$ and arbitrary t_0 for Gaussian random forces with zero mean. As an example, the motion of a spherical particle in an unbounded viscoelastic fluid with a particular relaxation time was considered. The velocity correlation function found after tedious calculations from the obtained FPE coincides in the case of stationary random process with the result that is easily calculated using the approach of the present work.

4. Hydrodynamic Brownian motion in the magnetic field

The description of the diffusion-like processes with the use of equation (1) is not the most general one. In some cases different phenomenological equations can possess a more appropriate description. As discussed in Introduction, the used method is not limited to (1). Here we shall demonstrate the efficiency of the Vladimirsky approach by the solution of a problem that naturally arises in the theory of the BM, namely, the BM with hydrodynamic memory [19]. This problem can be formulated using the LE as well. However, the Stokes friction force $-\gamma\upsilon$, which is valid only for the steady motion of the particle (at long times), should be replaced by a force that takes into account the history of the particle motion. This force appears as a consequence of the inertial effects in the motion of a particle in a liquid, and for incompressible fluids was first derived by Boussinesq [20]. The solution of this problem for BPs was given in the work [21] where, among others, it has been shown that the classic LE gives the correct results only for particles with the density much larger than the density of the surrounding fluid (and simultaneously at short times), or at long times when, however, the Einstein result is valid. Later this task was studied in a number of papers and exactly solved in Ref. [22] (for a review see [23 - 25]). With the approach used in the present work the solution can be obtained in a simpler way than so far. Here we will consider a more complicated problem of the movement of a charged BP in the magnetic field. For the motion along the field or in its absence the previous results [21, 22] will be recovered. The hydrodynamic motion of the BP across the field was studied already in the older paper [26]. Our solution corrects the results from Ref. [26] in several points.

The Boussinesq force on a spherical particle of radius R is

$$\mathbf{F}(t) = -\gamma\left\{\boldsymbol{\upsilon}(t) + \frac{\rho R^2}{9\eta}\frac{\mathrm{d}\boldsymbol{\upsilon}}{\mathrm{d}t} + \sqrt{\frac{\rho R^2}{\pi\eta}}\int_{-\infty}^{t}\frac{\mathrm{d}\boldsymbol{\upsilon}}{\mathrm{d}t'}\frac{\mathrm{d}t'}{\sqrt{t-t'}}\right\}, \tag{9}$$

where the friction factor $\gamma = 6\pi\eta R$, ρ is the density and η the viscosity of the solvent. This equation is valid for the times $t >> R/c$ (c is the sound velocity), *i.e.*, very short times when the compressibility effects play a role are excluded from the consideration. If Q is the charge of the particle of mass m and $\mathbf{B}$ is the constant induction of the magnetic field along the axis z, then the LE for the BP reads

$$m\dot{\boldsymbol{\upsilon}}(t) + \mathbf{F}(t) = Q\boldsymbol{\upsilon}\times\mathbf{B} + \boldsymbol{\eta}(t)\,. \tag{10}$$

The projection of this equation onto the axis z does not contain the magnetic force so that along the field we have the motion of a free BP. Equation (2) for $V_z = \mathrm{d}\xi_z/\mathrm{d}t$ with ξ_z being the MSD in the z direction can be written in the form

$$\dot{V}_z(t) + \frac{1}{\tau}\sqrt{\frac{\tau_R}{\pi}}\int_0^t \frac{\dot{V}_z(t')}{\sqrt{t-t'}}\mathrm{d}t' + \frac{1}{\tau}V_z(t) = \frac{2k_B T}{M}\,, \tag{11}$$

where $M = m + m_S/2$ (m_S is the mass of the solvent displaced by the particle). The characteristic times in this equation are $\tau = M/\gamma$ (the relaxation time of the usual BP) and $\tau_R = R^2\rho/\eta$ (the vorticity time). The Laplace transformation (11) gives

$$\tilde{V}_z(s) = \frac{2k_B T}{M} s^{-1}\left(s + \sqrt{\tau_R}\tau^{-1}s^{1/2} + \tau^{-1}\right)^{-1}. \tag{12}$$

The inverse transform is found after expanding the term $(...)^{-1}$ in simple fractions $(s - \lambda_{1,2})^{-1}$, where $\lambda_{1,2} = -\left(\sqrt{\tau_R}/2\tau\right)\left(1 \mp \sqrt{1-4\tau/\tau_R}\right)$ are the roots of equation $s + \sqrt{\tau_R}\tau^{-1}s^{1/2} + \tau^{-1} = 0$. Then

$$\tilde{V}_z(s) = \frac{2k_B T}{Ms}\frac{1}{\lambda_2 - \lambda_1}\left(\frac{1}{\sqrt{s}-\lambda_2} - \frac{1}{\sqrt{s}-\lambda_1}\right), \tag{13}$$

so that [14]

$$V_z(t) = \frac{2k_B T}{M}\frac{1}{\lambda_2 - \lambda_1}\left\{\frac{1}{\lambda_2}\left[\exp(\lambda_2^2 t)\mathrm{erfc}\left(-\lambda_2\sqrt{t}\right) - 1\right] - \frac{1}{\lambda_1}\left[\exp(\lambda_1^2 t)\mathrm{erfc}\left(-\lambda_1\sqrt{t}\right) - 1\right]\right\}. \tag{14}$$

The VAF is expressed by a similar equation, if one divides $V_z(t)$ by 2, in {...} replaces $1/\lambda_{1,2}$ with $\lambda_{1,2}$, and in both square brackets omit -1. This expression exactly corresponds to the solutions found in Refs. [21, 22] by different methods but differs from the solution [26] due to the difference in the roots $\lambda_{1,2}$. The solution at long times contains the so called long-time tails that became famous after their discovery in the computer experiments [27, 28]. For the VAF at $t \to \infty$ it follows from equation (14) that the longest-lived tail is $\sim t^{-3/2}$. The MSD is obtained by integrating this equation,

$$\xi_z(t) = 2D\left\{t - 2\left(\frac{\tau_R t}{\pi}\right)^{1/2} + \tau_R - \tau \right.$$
$$\left. + \frac{1}{\tau}\frac{1}{\lambda_2 - \lambda_1}\left[\frac{\exp\left(\lambda_2^2 t\right)}{\lambda_2^3}\mathrm{erfc}\left(-\lambda_2\sqrt{t}\right) - \frac{\exp\left(\lambda_1^2 t\right)}{\lambda_1^3}\mathrm{erfc}\left(-\lambda_1\sqrt{t}\right)\right]\right\}, \tag{15}$$

$$\xi_z(t) \approx 2Dt\left\{1 - 2\left(\frac{\tau_R}{\pi\tau}\right)^{\frac{1}{2}} + \frac{2}{9}\left(4 - \frac{m}{m_s}\right)\frac{\tau_R}{t} - \frac{1}{9\sqrt{\pi}}\left(7 - 4\frac{m}{m_s}\right)\left(\frac{\tau_R}{t}\right)^{\frac{3}{2}} + \ldots\right\}, \quad t \to \infty, \tag{16}$$

where $D = k_B T\tau/M$ is the diffusion coefficient of the particle. Equation (15) corresponds very well to experiment [29].

For the particle motion across the magnetic field we have from (10) two equations for the x and y components of the velocity [26] and, after the application of Vladimirsky rule, the equation for the quantity $V_x + V_y = \mathrm{d}\xi_{xy}/\mathrm{d}t$ that determine the MSD $\xi_{xy} = \xi_x + \xi_y$ of the particle in the plane perpendicular to the field [19]. In the Laplace transformation we obtain for $\tilde{V}_{xy}(s) = \Lambda\{V_x(t) + V_y(t)\}$

$$\tilde{V}_{xy}(s) = \frac{4k_B T}{Ms}\frac{\psi(s)}{\psi^2(s) + \tau_c^{-2}}, \tag{17}$$

where $\psi(s) = s+\tau^{-1}\sqrt{\tau_R s}+\tau^{-1}$ and the new characteristic time τ_c is connected to the cyclotron frequency $\omega_c = QB/M = 1/\tau_c$. The roots $\lambda_{1,2}$ of $\psi(s)$ have been determined after equation (12) and for the four roots of the denominator $\psi^2(s)+\tau_c^{-2}$ we have $2\kappa_{1,2} = \lambda_1+\lambda_2 \pm\sqrt{(\lambda_1-\lambda_2)^2-4\mathrm{i}\omega_c}$ and $2\kappa_{3,4} = \lambda_1+\lambda_2 \pm\sqrt{(\lambda_1-\lambda_2)^2+4\mathrm{i}\omega_c}$. Using them, the s-dependent fraction in (18) can be decomposed in simple fractions,

$$\frac{2\psi(s)}{\psi(s)^2+\tau_c^{-2}} = \frac{1}{\kappa_2-\kappa_1}\left(\frac{1}{\sqrt{s}-\kappa_2}-\frac{1}{\sqrt{s}-\kappa_1}\right)+\frac{1}{\kappa_4-\kappa_3}\left(\frac{1}{\sqrt{s}-\kappa_4}-\frac{1}{\sqrt{s}-\kappa_3}\right),$$

after which the Laplace transform tables can be used [14]. This yields

$$V_{xy}(t) = \frac{2k_BT}{M}\left\{\left(\frac{1}{\kappa_1\kappa_2}+\frac{1}{\kappa_2-\kappa_1}\left[f\left(\kappa_2,t\right)-f\left(\kappa_1,t\right)\right]\right)+\left(\kappa_1\to\kappa_3,\kappa_2\to\kappa_4\right)\right\}, \tag{18}$$

where $f(\kappa,t) = \kappa^{-1}\exp(\kappa^2 t)\mathrm{erfc}(-\kappa\sqrt{t})$. The MSD is readily obtained by the integration of $V_{xy}(t)$ from 0 to t. Using

$$\varphi(\kappa,t) = \int_0^t f(\kappa,t')\mathrm{d}t' = \frac{1}{\kappa^3}\left[\exp\left(\kappa^2 t\right)\mathrm{erfc}(-\kappa\sqrt{t})-1\right]-\frac{2}{\kappa^2}\sqrt{\frac{t}{\pi}},$$

one finds

$$\begin{aligned}\xi_{xy}(t) = \frac{2k_BT}{M}\Bigg\{&\left(\frac{t}{\kappa_1\kappa_2}+2\sqrt{\frac{t}{\pi}}\frac{\kappa_1+\kappa_2}{(\kappa_1\kappa_2)^2}+\frac{\kappa_1^2+\kappa_1\kappa_2+\kappa_2^2}{(\kappa_1\kappa_2)^3}\right.\\ &\left.+\frac{1}{\kappa_2-\kappa_1}\left[\frac{1}{\kappa_2^3}\exp\left(\kappa_2^2 t\right)\mathrm{erfc}\left(-\kappa_2\sqrt{t}\right)-\frac{1}{\kappa_1^3}\exp\left(\kappa_1^2 t\right)\mathrm{erfc}\left(-\kappa_1\sqrt{t}\right)\right]\right)\\ &+\left(\kappa_1\to\kappa_3,\kappa_2\to\kappa_4\right)\Bigg\}.\end{aligned} \tag{19}$$

In the absence of the field ($\omega_c = 0$) $\kappa_1 = \kappa_3 = \lambda_1$, $\kappa_2 = \kappa_4 = \lambda_2$, and $\xi_{xy} = 2\xi_z$, with ξ_z from (16).

To rewrite equation (19) in terms of the characteristic times τ, τ_c, and τ_R, one has to use κ_i shown after equation (17) and λ_i from (13). This gives $\kappa_1\kappa_2 = \tau^{-1}+\mathrm{i}\tau_c^{-1}$, $\kappa_3\kappa_4 = \tau^{-1}-\mathrm{i}\tau_c^{-1}$, $\kappa_1+\kappa_2 = \kappa_3+\kappa_4 = -\tau_R^{1/2}\tau^{-1}$, *etc*. For long times we use the expansion of the function erfc(z) for large arguments z [14], which, up to the longest-lived tail, gives

$$\xi_{xy}(t) \approx \frac{4k_BT}{M}\frac{\tau t}{1+\left(\tau/\tau_c\right)^2}\left(1-2\sqrt{\frac{\tau_R}{\pi t}}\frac{1-\left(\tau/\tau_c\right)^2}{1+\left(\tau/\tau_c\right)^2}+...\right), \quad t\to\infty, \tag{20}$$

with $\tau/\tau_c = MQB/(6\pi\eta Rm) = QB(1+\rho/2\rho')/(6\pi R)$. Here ρ and ρ' are the density of the fluid and the particle, respectively. Note that even the Einstein limit of equation (20) differs from the previous result [26] (the agreement is only when $M = m$ (or at $\omega_c = 0$)). At short times we have the ballistic motion (independent of the magnetic field) as from equation (2), $\xi_{xy}(t) \approx 2k_B T t^2 / M$, but with a different M, which is now $M = m + m_s/2$. This apparent contradiction with the equipartition theorem is just a consequence of the assumption of solvent incompressibility, due to which the limit $t \to 0$ cannot be accomplished. The correct result that contains only the mass of the particle, m, is achieved when the compressibility of the solvent is taken into account (on the time scale ~ 10^{-12} s that characterizes the collisions of the BP with the surrounding molecules the VAF $\Phi(t)$ decays from the equipartition value k_BT/m to $k_BT/(m + m_s/2)$ through the emission of sound waves). This is one more task that is straightforwardly solvable by the present method. Recently [30], the motion of a charged BP in a magnetic field was studied in the case when the aftereffect exponentially decreases in the time. Also this problem can be effectively solved by the proposed method [31].

5. Conclusions

In conclusion, we have described an efficient approach to the solution of the problems of diffusion with memory that are modeled using the generalized Langevin equation. The method consists in converting this stochastic integro-differential equation into a deterministic one for the mean square displacement. It allowed us to solve in a very simple way the specific problem when the Brownian particle is driven by an exponentially correlated stochastic force. The description of diffusion-like processes with the use of the GLE is not the most general one. In many cases different phenomenological equations could possess a more appropriate model. It is thus important that the used method is not limited to the GLE equations. We have demonstrated it on the problem of the hydrodynamic Brownian motion of charged particles in the magnetic field. Other applications that are currently of great interest concern, for example, the motion of Brownian particles dragged by optical tweezers [32, 33], the anomalous motion of colloidal particles under the influence of various external fields [34, 35], the behavior of mesoscale electric circuits in contact with the thermal bath [36, 37], the motion of magnetic domain walls [38], or the dynamics of polymers [24, 25, 39 - 41]. Among the mentioned problems those close to the tasks considered in the present work relate to the Brownian motion in the presence of magnetic field [42, 43]. In Ref. [42], the motion of a charged Brownian particle in a harmonic trap was theoretically described in the case when the trap is dragged with a constant velocity and when the particle is subjected to an ac force. The work distribution of the particle has been found analytically and the Jarzynski equality [44, 45] was verified. In the recent paper [43], a similar problem was considered for a trapped particle under the action of a constant magnetic field and a time-dependent electric field. The main aim of that work was to prove the validity of the stationary state fluctuation theorem [46] and the transient fluctuation theorem [47]. Note that the equations of motion for the particle were chosen in the form of the usual Langevin equations with the Stokes friction and the white noise force. It would be interesting to develop these works to the case of the GLE with hydrodynamic memory, which is a suitable model if the Brownian particle moves in a liquid. In our opinion, future effort should be also oriented on the generalization of the proposed method to situations when quantum effects are significant [36], and to overcome its restriction to linear models.

Acknowledgments. This work was supported by the Agency for the Structural Funds of the EU within the projects NFP 26220120021 and 26220120033, and by the grant VEGA 1/0300/09.

References

[1] R. Klages, G. Radons, I. M. Sokolov (Eds.), Anomalous Transport: Foundations and Applications, Wiley-VCH, Berlin, 2008.
[2] A. Einstein, Ann. Phys. 17 (1905) 549.
[3] R. Kubo, Rep. Prog. Phys. 29 (1966) 255.
[4] W. T. Coffey, Yu. P. Kalmykov, J. T. Waldron, The Langevin Equation, World Scientific, New Jersey, 2005.
[5] P. Langevin, C. R. Acad. Sci. Paris 146 (1908) 530.
[6] V. V. Vladimirsky, Zhur. Ekper. Teor. Fiz. 12 (1942) 199 (in Russian).
[7] Yu. L. Klimontovich, Statistical Theory of Open Systems, Vol. 1, Chapters 19 and 20, Kluwer Academic Publ., Dordrecht, 1995.
[8] R. M. Mazo, Brownian Motion. Fluctuations, Dynamics, and Applications, Oxford University Press, New York, 2009.
[9] V. Balakrishnan, Pramana 12 (1979) 301.
[10] P. P. J. M. Schram, I. P. Yakimenko, Physica A 260 (1998) 73.
[11] R. Morgado, F. A. Oliveira, G. G. Batrouni, A. Hansen, Phys. Rev. Lett. 89 (2002) 100601.
[12] L. C. Lapas, R. Morgado, M. H. Vainstein, J. M. Rubi, F. A. Oliveira, Phys. Rev. Lett. 101 (2008) 230602.
[13] J. Tothova, G. Vasziova, L. Glod, V. Lisy, Eur. J. Phys. 32 (2011) 645.
[14] M. Abramowitz, I. A. Stegun, Handbook of Mathematical Functions, National Bureau of Standards, Washington, DC, 1964.
[15] I. Goychuk, Phys. Rev. E 80 (2009) 046125.
[16] H. Risken, The Fokker-Planck Equation, Springer-Verlag, Berlin, 1989.
[17] R. F. Fox, J. Math. Phys. 18 (1976) 2331.
[18] V. S. Volkov, V. N. Pokrovsky, J. Math. Phys. 24 (1983) 257.
[19] J. Tothova, V. Lisy, Proc. 14th Czech and Slovak Conf. on Magnetism, July 6-9, 2010, Kosice, Slovakia (Special issue of Acta Physica Polonica 118 (2010) 1051).
[20] J. Boussinesq, C. R. Acad. Sci. Paris 100 (1885) 935.
[21] V. Vladimirsky, Ya. Terletzky, Zhur. Eksp. Teor. Fiz. 15 (1945) 259 (in Russian).
[22] E. J. Hinch, J. Fluid. Mech. 72 (1975) 499.
[23] V. Lisy, J. Tothova, arXiv:cond-mat/0410222v1 [cond-mat.stat-mech]
[24] J. Tothova, V. Lisy V, A. V. Zatovsky, J. Chem. Phys. 119 (2003) 13135.
[25] V. Lisy, J. Tothova J, A. V. Zatovsky, J. Chem. Phys. 121 (2004) 10699.
[26] Karmeshu, J. Phys. Soc. Jpn. 34 (1973) 1467.
[27] A. Rahman, Phys. Rev. A 136 (1964) 405.
[28] B. J. Alder, T. E. Wainwright, Phys. Rev. Lett. 18 (1967) 988.
[29] B. Lukic, S. Jeney, C. Tischer, A. J. Kulik, L. Forro, E. L. Florin, Phys. Rev. Lett. 95 (2005) 160601.
[30] F. N. C. Paraan, M. P. Solon, J. P. Esguerra, Phys. Rev. E 77 (2008) 022101.
[31] V. Lisy, J. Tothova, 2011 arXiv:1107.1891v1 [cond-mat.soft].
[32] R. Van Zon, E. G. D. Cohen, Phys. Rev. E 67 (2003) 046102.
[33] T. Li, S. Kheifets, D. Medellin, M. G. Raizen, Science 328 (2010) 1673.
[34] D. M. Carberry, J. C. Reid, G. M. Wang, E. M. Sevick, D. J. Searles, D. J. Evans, Phys. Rev. Lett. 92 (2004) 140601.
[35] H. Lowen, J. Phys.: Condens. Matter 21 (2009) 474203.
[36] A. E. Allahverdyan, Th. M. Nieuwenhuizen, Phys. Rev. B 66 (2002) 115309.
[37] N. Garnier, S. Ciliberto, Phys. Rev. E 71 (2005) 060101(R).
[38] E. Saitoh, H. Miyajima, T. Yamaoka, G. Tatara, Nature 432 (2004) 203.
[39] D. Panja, J. Stat. Mech.: Theory and Experiment, L02001 (2010) 8 pp.

[40] D. Panja, J. Stat. Mech.: Theory and Experiment P06011 (2010) 34 pp.
[41] V. Lisy, J. Tothova, A. V. Zatovsky, J. Stat. Mech.: Theory and Experiment P01024 (2008) 12 pp.
[42] A. M. Jayannavar, M. Sahoo, Phys. Rev. E 75 (2007) 032102.
[43] J. I. Jimenez-Aquino, F. J. Uribe, R. M. Velasco, J. Phys. A: Math. Theor. 43 (2010) 255001.
[44] C. Jarzynski, Phys. Rev. Lett. 78 (1997) 2690.
[45] C. Jarzynski, Phys. Rev. E 56 (1997) 5018.
[46] D. J. Evans, E. G. D. Cohen, G. P. Morris, Phys. Rev. Lett. 71 (1993) 2401.
[47] D. J. Evans, D. J. Searles, Phys. Rev. E 50 (1994) 1645.