diffusion-fundamentals

The Open-Access Journal for the Basic Principles of Diffusion Theory, Experiment and Application

Anomalous conductance and diffusion in complex networks

Shlomo Havlin [1,2], *Eduardo López* [2,3], *Sergey V. Buldyrev* [2,4], *and H. Eugene Stanley* [2]

[1] Minerva Center & Department of Physics, Bar-Ilan University, Israel
[2] Center for Polymer Studies, Boston University, USA
[3] Theoretical Division, Los Alamos National Laboratory, USA
[4] Department of Physics, Yeshiva University New York, USA

Abstract

We study transport properties such as conductance and diffusion of complex networks such as scale-free and Erdős-Rényi networks. We consider the equivalent conductance G between two arbitrarily chosen nodes of random scale-free networks with degree distribution $P(k) \sim k^{-\lambda}$ and Erdős-Rényi networks in which each link has the same unit resistance. Our theoretical analysis for scale-free networks predicts a broad range of values of G (or the related diffusion constant D), with a power-law tail distribution $\Phi_{\rm SF}(G) \sim G^{-g_G}$, where $g_G = 2\lambda - 1$. We confirm our predictions by simulations of scale-free networks solving the Kirchhoff equations for the conductance between a pair of nodes. The power-law tail in $\Phi_{\rm SF}(G)$ leads to large values of G, thereby significantly improving the transport in scale-free networks, compared to Erdős-Rényi networks where the tail of the conductivity distribution decays exponentially. Based on a simple physical "transport backbone" picture we suggest that the conductances of scale-free and Erdős-Rényi networks can be approximated by $ck_Ak_B/(k_A + k_B)$ for any pair of nodes A and B with degrees k_A and k_B. Thus, a single parameter c characterizes transport on both scale-free and Erdős-Rényi networks.

1 Introduction

Diffusion in many random structures is "anomalous," i.e., fundamentally different than that in regular space [1, 2, 3]. The anomaly is due to the random substrate on which diffusion is constrained to take place. Random structures are found in many places in the real world, from oil reservoirs to the Internet, making the study of anomalous diffusion properties a far-reaching field. In this problem, it is paramount to relate the structural properties of the medium with the diffusion properties.

An important and recent example of random substrates is that of complex networks. Research on this topic has uncovered their importance for real-world problems as diverse as the World Wide Web and the Internet to cellular networks and sexual-partner networks [4].

Two distinct models describe the two limiting cases for the structure of the complex networks. The first of these is the classic Erdős-Rényi model of random networks [5], for which sites are connected with a link with probability p and disconnected (no link) with probability $1-p$ (see Fig. 1). In this case, the degree distribution (distribution of the number of connections of a link) is a Poisson distribution

$$P(k) \sim \frac{(\overline{k})^k e^{-\overline{k}}}{k!}, \tag{1}$$

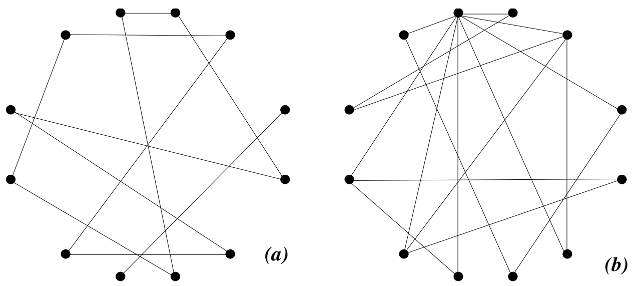

Figure 1: (a) Schematic of an Erdős-Rényi network of $N = 12$ and $p = 1/6$. Note that in this example ten nodes have $k = 2$ connections, and two nodes have $k = 1$ connections. This illustrates the fact that for Erdős-Rényi networks, the range of values of degree is very narrow, typically close to $\overline{k}$. (b) Schematic of a scale-free network of $N = 12$, $k_{\min} = 2$ and $\lambda \approx 2$. We note the presence of a hub with $k_{\max} = 8$ which is connected to many of the other links of the network.

where $\overline{k} \equiv \sum_{k=1}^{\infty} kP(k)$ is the average degree of the network. Mathematicians discovered critical phenomena through this model. For instance, just as in percolation on lattices, there is a critical value $p = p_c$ above which the largest connected component of the network has a mass that scales with the system size N, but below p_c, there are only small clusters of the order of $\log N$. Another charateristic of an Erdős-Rényi network is its "small-world" property which means that the average distance d (or diameter) between all pairs of nodes of the network scales as $\log N$ [6]. The other model, recently identified as the characterizing topological structure of many real world systems, is the Barabási-Albert scale-free network [7], characterized by a scale-free degree distribution:

$$P(k) \sim k^{-\lambda} \qquad [k_{\min} \leq k \leq k_{\max}], \tag{2}$$

The cutoff value $k_{\min}$ represents the minimum allowed value of k on the network ($k_{\min} = 2$ here), and $k_{\max} \equiv k_{\min} N^{1/(\lambda-1)}$, the typical maximum degree of a network with N nodes [8, 9]. The scale-free feature allows a network to have some nodes with a large number of links ("hubs"), unlike the case for the Erdős-Rényi model of random networks [5, 6]. Scale-free networks with $\lambda > 3$ have $d \sim \log N$, while for $2 < \lambda < 3$ they are "ultra-small-world" since the diameter scales as $d \sim \log \log N$ [4, 8].

Here we review our recent study of transport in complex networks [10]. We find that for scale-free networks with $\lambda \geq 2$, transport properties characterized by conductance display a power-law tail distribution that is related to the degree distribution $P(k)$. The origin of this power-law tail is due to pairs of nodes of high degree which have high conductance. Thus, transport in scale-free networks is better than in Erdős-Rényi random networks since the high degree nodes carry much of the traffic in the network. Also, we present a simple physical picture of transport in scale-free and Erdős-Rényi networks and test it through simulations. The results of our study are relevant to problems of diffusion in scale-free networks, given that conductivity and diffusivity are related by the Einstein relation [1, 2, 3]. Due to the exponential decay of the degree distribution, Erdős-Rényi networks lack hubs and their properties, including transport, are controlled mainly by the average degree $\overline{k}$. [6, 11].

2 Transport in complex networks

Most of the work done so far regarding complex networks has concentrated on static topological properties or on models for their growth [4, 8, 12, 13]. Transport features have not been extensively studied with the exception of random walks on specific complex networks [14, 15, 16]. Transport properties are important because they contain information about network function [17]. Here, we study the electrical conductance G between two nodes A and B of Erdős-Rényi and scale-free networks when a potential difference is imposed between them. We assume that all the links have equal resistances of unit value [18].

To construct an Erdős-Rényi network, we begin with N nodes and connect each pair with probability p. To generate a scale-free network with N nodes, we use the Molloy-Reed algorithm [19], which allows for the construction of random networks with arbitrary degree distribution. We generate k_i copies of each node i, where the probability of having k_i satisfies $P(k_i) \sim k_i^{-\lambda}$. We then randomly pair these copies of the nodes in order to construct the network, making sure that two previously-linked nodes are not connected again, and also excluding links of a node to itself [20].

We calculate the conductance G of the network between two nodes A and B using the Kirchhoff method, [21], where entering and exiting potentials are fixed to $V_A = 1$ and $V_B = 0$. We solve a set of linear equations to determine the potentials V_i of all nodes of the network. Finally, the total current $I \equiv G$ entering at node A and exiting at node B is computed by adding the outgoing currents from A to its nearest neighbors through $\sum_j (V_A - V_j)$, where j runs over the neighbors of A.

First, we analyze the probability density function (pdf) $\Phi(G)$ which comes from $\Phi(G)dG$, the probability that two nodes on the network have conductance between G and $G + dG$. To this end, we introduce the cumulative distribution $F(G) \equiv \int_G^\infty \Phi(G')dG'$, shown in Fig. 2(a) for the Erdős-Rényi and scale-free ($\lambda = 2.5$ and $\lambda = 3.3$, with $k_{\min} = 2$) cases. We use the notation $\Phi_{\rm SF}(G)$ and $F_{\rm SF}(G)$ for scale-free, and $\Phi_{\rm ER}(G)$ and $F_{\rm ER}(G)$ for Erdős-Rényi. The function $F_{\rm SF}(G)$ for both $\lambda = 2.5$ and 3.3 exhibits a tail region well fit by the power law

$$F_{\rm SF}(G) \sim G^{-(g_G - 1)}, \qquad (3)$$

and the exponent $(g_G - 1)$ increases with λ. In contrast, $F_{\rm ER}(G)$ decreases exponentially with G.

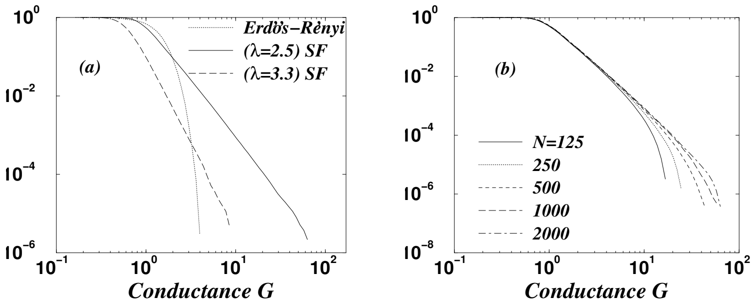

Figure 2: (a) Comparison for networks with $N = 8000$ nodes between the cumulative distribution functions of conductance for the Erdős-Rényi and the scale-free cases (with $\lambda = 2.5$ and 3.3). Each curve represents the cumulative distribution $F(G)$ vs. G. The simulations have at least 10^6 realizations. (b) Effect of system size on $F_{\rm SF}(G)$ vs. G for the case $\lambda = 2.5$. The cutoff value of the maximum conductance $G_{\max}$ progressively increases as N increases.

Increasing N does not significantly change $F_{\rm SF}(G)$ (Fig. 2(b)) except for an increase in the upper cutoff $G_{\max}$, where $G_{\max}$ is the typical maximum conductance, corresponding to the value of G at which $\Phi_{\rm SF}(G)$ crosses over from a power law to a faster decay. We observe no change of the exponent g_G with N. The increase of $G_{\max}$ with N implies that the average conductance $\overline{G}$ over all pairs also increases slightly [22].

We next study the origin of the large values of G in scale-free networks and obtain an analytical relation between λ and g_G. Larger values of G require the presence of many parallel paths, which we hypothesize arise from the high degree nodes. Thus, we expect that if either of the degrees k_A or k_B of the entering and exiting nodes is small (e.g. $k_A > k_B$), the conductance G between A and B is small since there are at most k different parallel branches coming out of a node with degree k. Thus, a small value of k implies a small number of possible parallel branches, and therefore a small value of G. To observe large G values, it is therefore necessary that both k_A and k_B be large.

We test this hypothesis by large scale computer simulations of the conditional pdf $\Phi_{\rm SF}(G|k_A, k_B)$ for specific values of the entering and exiting node degrees k_A and k_B. Consider first $k_B \ll k_A$, and the effect of increasing k_B, with k_A fixed. We find that $\Phi_{\rm SF}(G|k_A, k_B)$ is narrowly peaked (Fig. 3(a)) so that it is well characterized by G^*, the value of G when $\Phi_{\rm SF}$ is a maximum. We find similar results for Erdős-Rényi networks. Further, for increasing k_B, we find [Fig. 3(b)] G^* increases as $G^* \sim k_B^\alpha$, with $\alpha = 0.96 \pm 0.05$ consistent with the possibility that as $N \to \infty$, $\alpha = 1$ which we assume henceforth.

For the case of $k_B \gtrsim k_A$, G^* increases less fast than k_B, as can be seen in Fig. 3(c) where we plot G^*/k_B against the scaled degree $x \equiv k_A/k_B$. The collapse of G^*/k_B for

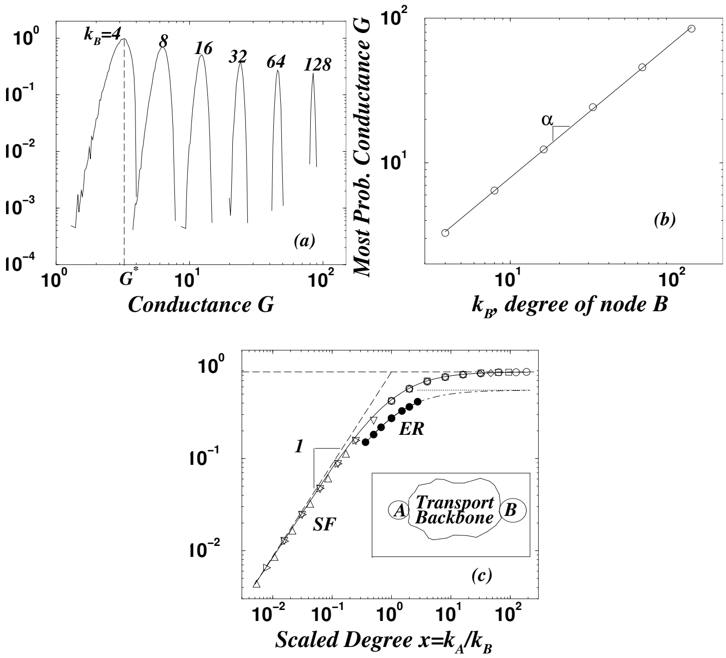

Figure 3: (a) The pdf $\Phi_{\rm SF}(G|k_A,k_B)$ vs. G for $N=8000$, $\lambda=2.5$ and $k_A=750$ (k_A is close to the typical maximum degree $k_{\max}=800$ for $N=8000$). (b) Most probable values G^*, estimated from the maxima of the distributions in Fig. 3(a), as a function of the degree k_B. The data support a power law behavior $G^*\sim k_B^\alpha$ with $\alpha=0.96\pm0.05$. (c) Scaled most probable conductance G^*/k_B vs. scaled degree $x\equiv k_A/k_B$ for system size $N=8000$ and $\lambda=2.5$, for several values of k_A and k_B: $\square$ ($k_A=8$, $8\le k_B\le 750$), $\diamondsuit$ ($k_A=16$, $16\le k_B\le 750$), $\triangle$ ($k_A=750$, $4\le k_B\le 128$), $\bigcirc$ ($k_B=4$, $4\le k_A\le 750$), $\bigtriangledown$ ($k_B=256$, $256\le k_A\le 750$), and $\triangleright$ ($k_B=500$, $4\le k_A\le 128$). The curve crossing the symbols is the predicted function $G^*/k_B=f(x)=cx/(1+x)$ obtained from Eq. (7). We also show G^*/k_B vs. scaled degree $x\equiv k_A/k_B$ for Erdős-Rényi networks with $\overline{k}=2.92$, $4\le k_A\le 11$ and $k_B=4$ (symbol $\bullet$). The curve crossing the symbols represents the theoretical result according to Eq. (7), and an extension of this line to represent the limiting value of G^*/k_B (dotted-dashed line). The probability to obtain $k_A>11$ is extremely small in Erdős-Rényi networks, and thus we are unable to obtain significant statistics. Scaling function $f(x)$, as seen here, exhibits a crossover from a linear behavior to the constant c ($c=0.87\pm0.02$ for scale-free networks, horizontal dashed line, and $c=0.55\pm0.01$ for Erdős-Rényi, dotted line). The inset shows a schematic of the "transport backbone" picture, where the circles labeled A and B denote nodes A and B and their associated links which do not belong to the "transport backbone".

different values of k_A and k_B indicates that G^* scales as

$$G^* \sim k_B f\left(\frac{k_A}{k_B}\right). \tag{4}$$

Below we study the possible origin of this function.

3 Transport backbone picture

The behavior of the scaling function $f(x)$ can be interpreted using the following simplified "transport backbone" picture [Fig. 3(c) inset], for which the effective conductance G between nodes A and B satisfies

$$\frac{1}{G} = \frac{1}{G_A} + \frac{1}{G_{tb}} + \frac{1}{G_B}, \tag{5}$$

where $1/G_{tb}$ is the resistance of the "transport backbone" while $1/G_A$ (and $1/G_B$) are the resistances of the set of bonds near node A (and node B) not belonging to the "transport backbone". It is plausible that G_A is linear in k_A, so we can write $G_A = ck_A$. Since node B is equivalent to node A, we expect $G_B = ck_B$. Hence

$$G = \frac{1}{1/ck_A + 1/ck_B + 1/G_{tb}} = k_B \frac{ck_A/k_B}{1 + k_A/k_B + ck_A/G_{tb}}, \tag{6}$$

so the scaling function defined in Eq. (4) is

$$f(x) = \frac{cx}{1 + x + ck_A/G_{tb}} \approx \frac{cx}{1+x}. \tag{7}$$

The second equality follows if there are many parallel paths on the "transport backbone" so that $1/G_{tb} \ll 1/ck_A$ [23]. The prediction (7) is plotted in Fig. 3(c) for both scale-free and Erdős-Rényi networks and the agreement with the simulations supports the approximate validity of the transport backbone picture of conductance in scale-free and Erdős-Rényi networks.

The agreement of (7) with simulations has a striking implication: the conductance of a scale-free and Erdős-Rényi network (scale-free and Erdős-Rényi) depends on only one parameter c. Further, since the distribution of Fig. 3(a) is sharply peaked, a single measurement of G for any values of the degrees k_A and k_B of the entrance and exit nodes suffices to determine G^*, which then determines c and hence through Eq. (7) the conductance for all values of k_A and k_B.

Within this "transport backbone" picture, we can analytically calculate $F_{\mathrm{SF}}(G)$. Using Eq. (4), and the fact that $\Phi_{\mathrm{SF}}(G|k_A, k_B)$ is narrow, yields [24]

$$\Phi_{\mathrm{SF}}(G) \sim \int P(k_B) dk_B \int P(k_A) dk_A \delta\left[k_B f\left(\frac{k_A}{k_B}\right) - G\right], \tag{8}$$

where $\delta(x)$ is the Dirac delta function. Performing the integration of Eq. (8) using (7), we obtain for $G < G_{\max}$

$$\Phi_{\rm SF}(G) \sim G^{-g_G} \qquad [g_G = 2\lambda - 1]. \tag{9}$$

Hence, for $F_{\rm SF}(G)$, we have $F_{\rm SF}(G) \sim G^{-(2\lambda-2)}$. To test this prediction, we perform simulations for scale-free networks and calculate the values of $g_G - 1$ from the slope of a log-log plot of the cumulative distribution $F_{\rm SF}(G)$. From Fig. 4(b) we find that

$$g_G - 1 = (1.97 \pm 0.04)\lambda - (2.01 \pm 0.13). \tag{10}$$

Thus, the measured slopes are consistent with the theoretical values predicted by Eq. (9) [25].

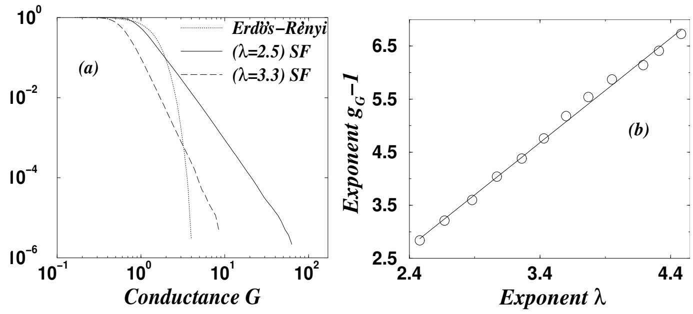

Figure 4: (a) Simulation results for the cumulative distribution $F_{\rm SF}(G)$ for λ between 2.5 and 3.5, consistent with the power law $F_{\rm SF} \sim G^{-(g_G-1)}$ (cf. Eq. (9)), showing the progressive change of the slope $g_G - 1$. (b) The exponent $g_G - 1$ from simulations (circles) with $2.5 < \lambda < 4.5$; shown also is a least square fit $g_G - 1 = (1.97 \pm 0.04)\lambda - (2.01 \pm 0.13)$, consistent with the predicted expression $g_G - 1 = 2\lambda - 2$ [cf. Eq. (9)].

The transport backbone conductance G_{tb} of scale-free networks can also be studied through its pdf $\Psi_{\rm SF}$ (see Fig. 5). To determine G_{tb}, we consider the contribution to the conductance of the part of the network with paths between A and B, excluding the contributions from the vicinities of nodes A and B, which are determined by the parameter c. The most relevant feature in Fig. 5 is that, for a given λ value, both $\Psi_{\rm SF}$ and $\Phi(G)$ have equal decay exponents, suggesting that they are also related to λ as Eq. (10). Figure 5 also shows that the values of G_{tb} are significantly larger than G.

4 Discussion

Next, we consider some further implications of our work. Our results show that larger values of G are found in scale-free networks with a much larger probability than in Erdős-Rényi networks, which raises the question if scale-free networks have better transport than

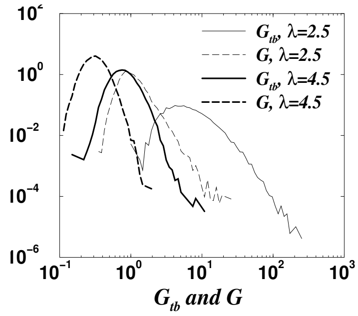

Figure 5: Comparison of pdf $\Psi(G_{tb})$ and $\Phi(G)$ for networks of $N = 8000$ for two values of λ.

Erdős-Rényi networks. To answer this question, we consider the average conductance between all the pairs of nodes in the network, which quantifies how efficient is the transport. However, since scale-free networks are heterogeneous in their degree, we must find a way to assign proper weights to the nodes. Recent work [26, 27, 28] suggests that in certain real-world networks, e.g. World-Ariline-Network [26] and biological networks [27], the conductances of links between nodes i and j are characterized by $(k_i k_j)^\beta$, with $\beta = 1/2$. Assuming this weight, and comparing scale-free and Erdős-Rényi networks with the same values of average degree $\overline{k}$ [29], we find that the average conductance of scale-free networks is larger than that of Erdős-Rényi networks (Table 1). Even larger average conductance for scale-free networks compared to Erdős-Rényi networks (Table 1) is obtained if one assumes [14] $\beta = 1$, i.e., that transport occurs with frequency proportional to the degree of the node. The case of $\beta = 0$ represents a "democratic" average, where all the pairs of nodes A and B are given the same weight. This case, which is not justified for heterogeneous networks, yields average conductance values for scale-free networks close to those of Erdős-Rényi networks (Table 1). In many real-world systems, degree dependent link conductances and frequent use of high degree nodes both occur, making transport on scale-free networks even more efficient than transport on Erdős-Rényi networks.

Finally, we point out that our study needs to be extended further. For instance, it has been found recently that many real-world scale-free networks posses fractal properties [30]. However, random scale-free and Erdős-Rényi networks, which are the subject of this study, do not display fractality. Since fractal substrates also lead to anomalous transport [1, 2, 3], it would be interesting to explore the effect of fractality on diffusion and conductance in fractal networks. This case is expected to have anomalous effects due to both the heterogeneity of the degree distribution and to the fractality of the network. Another interesting feature that should be studied is the effect on conductivity and diffusion of the correlation between distance of two nodes and their degree [31].

scale-free		$\beta = 1$	$\beta = 1/2$	$\beta = 0$
λ	$\overline{k}$	$\overline{G}_{SF}$ ($\overline{G}_{ER}$)	$\overline{G}_{SF}$ ($\overline{G}_{ER}$)	$\overline{G}_{SF}$ ($\overline{G}_{ER}$)
2.5	5.3	5.5 (2.1)	2.4 (2.0)	1.3 (1.9)
2.7	4.3	2.7 (1.5)	1.8 (1.5)	1.1 (1.4)
2.9	3.7	1.7 (1.2)	1.4 (1.2)	0.9 (1.1)
3.1	3.4	1.3 (1.0)	1.1 (0.9)	0.8 (0.9)
3.3	3.1	1.0 (0.9)	1.0 (0.8)	0.7 (0.7)
3.5	2.9	0.8 (0.7)	0.8 (0.7)	0.6 (0.7)

Table 1: Values of average conductance of scale-free and Erdős-Rényi networks for link weights defined as $(k_i k_j)^\beta$. In parenthesis we have indicated the values of the corresponding Erdős-Rényi networks.

5 Summary

In summary, we find that the conductance of scale-free networks is highly heterogeneous, and depends strongly on the degree of the two nodes A and B. Our results suggest that the diffusion constants are also heterogeneous in these networks, and depend on the degrees of the starting and ending nodes. We also find a power-law tail for $\Phi_{SF}(G)$ and relate the tail exponent g_G to the exponent λ of the degree distribution $P(k)$. This power law behavior makes scale-free networks better for tranport. Our work is consistent with a simple physical picture of how transport takes place in scale-free and Erdős-Rényi networks. This, so called "transport backbone" picture consists of the nodes A and B and their vicinities, and the rest of the network, which consititutes the transport backbone. Because of the great number of parallel paths contained in the transport backbone, transport takes place inside with very small resistance, and therefore the dominating effect of resistance comes from the vicinity of the node (A or B) with the smallest degree.

We thank the Office of Naval Research, the Israel Science Foundation, and the European NEST project DYSONET for financial support, and L. Braunstein, S. Carmi, R. Cohen, E. Perlsman, G. Paul, S. Sreenivasan, T. Tanizawa, and Z. Wu for discussions.

References

[1] S. Havlin and D. ben-Avraham, Adv. Phys. **36**, 695 (1987).

[2] D. ben-Avraham and S. Havlin *Diffusion and reactions in fractals and disordered systems* (Cambridge, New York, 2000).

[3] A. Bunde and S. Havlin, eds. *Fractals and Disordered Systems* (Springer, New York, 1996).

[4] R. Albert and A.-L. Barabási. Rev. Mod. Phys. **74**, 47 (2002); R. Pastor-Satorras and A. Vespignani, *Structure and Evolution of the Internet: A Statistical Physics*

Approach (Cambridge University Press, Cambridge, 2004); S. N. Dorogovsetv and J. F. F. Mendes, *Evolution of Networks: From Biological Nets to the Internet and WWW* (Oxford University Press, Oxford, 2003).

[5] P. Erdős and A. Rényi, Publ. Math. (Debreccen), **6**, 290 (1959).

[6] B. Bollobás, *Random Graphs* (Academic Press, Orlando, 1985).

[7] A.-L. Barabási and R. Albert, Science **286**, 509 (1999).

[8] R. Cohen, K. Erez, D. ben-Avraham, and S. Havlin, Phys. Rev. Lett. **85**, 4626 (2000).

[9] In principle, a node can have a degree up to $N - 1$, connecting to all other nodes of the network. The results presented here correspond to networks with upper cutoff $k_{\max} = k_{\min} N^{1/(\lambda-1)}$ imposed. We also studied networks for which $k_{\max}$ is not imposed, and found no significant differences in the pdf $\Phi_{\rm SF}(G)$.

[10] E. López, S. V. Buldyrev, S. Havlin, and H. E. Stanley, *Phys. Rev. Lett.* **94**, (2005).

[11] G. R. Grimmett and H. Kesten, J. Lond. Math. Soc. **30**, 171 (1984); math/0107068.

[12] P. L. Krapivsky, S. Redner, and F. Leyvraz, Phys. Rev. Lett. **85**, 4629 (2000).

[13] Z. Toroczkai and K. Bassler, Nature **428**, 716 (2004).

[14] J. D. Noh and H. Rieger, Phys. Rev. Lett. **92**, 118701 (2004).

[15] V. Sood, S. Redner, and D. ben-Avraham, J. Phys. A, **38**, 109 (2005).

[16] L. K. Gallos, Phys. Rev. E **70**, 046116 (2004).

[17] The dynamical properties we study are related to transport on networks and differ from those which treat the network topology itself as evolving in time [7, 12].

[18] The study of community structure in social networks has led some authors (M. E. J. Newman and M. Girvan, Phys. Rev. E **69**, 026113 (2004), and F. Wu and B. A. Huberman, Eur. Phys. J. B **38**, 331 (2004)) to develope methods in which networks are considered as electrical networks in order to identify communities. In these studies, however, transport properties have not been addressed.

[19] M. Molloy and B. Reed, Random Struct. Algorithms **6**, 161 (1995).

[20] We performed simulations with the node copies randomly matched, and also matched in order of degree from highest to lowest and obtained similar results.

[21] G. Kirchhoff, Ann. Phys. Chem. **72**, 497 (1847). ; N. Balabanian, *Electric Circuits* (McGraw-Hill, New York, 1994).

[22] The average conductance $\overline{G}$ of networks depends on the weight given to different pairs of nodes (see discussion after Eq. (10) and Table 1).

[23] Flux starts at node A, controlled by the conductance of the bonds in the vicinity of A. This flux passes into the "transport backbone", which is comprised of many parallel paths and hence has a high conductance. Finally, flux ends at node B, controlled by the conductance of the bonds in the vicinity of B (see inset of Fig. 3c). This is similar to traffic around a major freeway. Most of the limitations to transport occur in getting to the freeway ("node A") and then after leaving it ("node B"), but flow occurs easily on the freeway ("transport backbone").

[24] N. G. Van Kampen, *Stochastic Processes in Physics and Chemistry* (North-Holland, Amsterdam, 1992).

[25] The λ values explored here are limited by computer time considerations. In principle, we are unaware of any theoretical reason that would limit the validity of our results to any particular range of values of λ.

[26] A. Barrat, M. Barthélemy, R. Pastor-Satorras, and A. Vespignani, *Proc. Nat. Acad. Sci.* **101**, 3747 (2004).

[27] E. Almas, S. Redner, and P. Krapivsky, Phys. Rev. E **71**, 036124 (2005).

[28] K.-I. Goh, B. Kahng, and D. Kim, cond-mat/0410078.

[29] The cost of a network is defined as its number of links. Thus, two networks with equal N and $\overline{k}$ have equal cost. We compare each scale-free network with its correponding equal cost Erdős-Rényi network.

[30] C. M. Song, H. A. Makse, and S. Havlin, *Nature* **433**, 392 (2005).

[31] J. A. Holyst, J. Sienkiewicz, A. Fronczak, P.Fronczak, and K. Suchecki, *Physica A* **351**, 167 (2005).